NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1406

ON THE USE OF THE HARMONIC LINEARIZATION METHOD IN THE AUTOMATIC CONTROL THEORY

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By E. P. Popov

The method of harmonic linearization (harmonic balance), first proposed by N. M. Krylov and N. N. Bogolyubov (ref. 1) for the approximate investigation of nonlinear vibrations, has been developed and reeived wide practical application to problems in the theory of automatic control (refs. 3 to 6). Recently, some doubt has been expressed on the legitimacy of application of the method to these problems, and assertions were made on the absence in them of a small parameter of any kind. Nevertheless, the method gives practical, acceptable results and is a simple and powerful means in engineering computatations. Hence, the importance of questions arises as to its justification. The underlying principle of the method is the replacement of the given nonlinear equation by a linear equation. In establishing the method, a small parameter is considered whose presence makes it possible to speak, with some degree of approximation, of the solution of this new equation to the solution of the given nonlinear equation. In an article by the author (ref. 7), certain considerations were given on the presence of the small parameter, but this question has not as yet received a final answer. In the present report, a somewhat different approach to the problem is applied that permits: (a) establishing, in the clearest manner, the form of the presence of the small parameter in nonlinear problems of control theory, solvable by the method of harmonic linearization; (b) connecting it with previous intuitive physical concepts (with the "filter property") and extending the class of problems possessing this property; and (c) discussing various generalizations of the method.

The free motion (transition process and autovibration) for a very wide class of nonlinear systems of automatic control (ref. 7) are described by differential equations of the form

$$Q(p)x + R(p)F(x,px) = 0 \quad \left(p \equiv \frac{d}{dt}\right) \tag{1}$$

where $Q(p)$ and $R(p)$ are polynomials of any degree, of which some required properties will be established in the following paragraphs, and $F(x,px)$ is a given nonlinear function possible only with respect to assumptions of the most general character. However, in problems of the theory of control, no assumptions must be made as to the smallness of the nonlinear function $F(x,px)$ or to its small difference from a linear function. In order to render explicit the form in which it would be possible to write a small parameter in equation (1), we shall proceed as follows.

Let equation (1) have a periodic solution or a solution approximately periodic differing slightly from the sinusoidal. We write this solution in the form

$$x = x^* + \varepsilon y(t) \qquad x^* = a \sin \omega t \tag{2}$$

where ε denotes a small parameter, and $y(t)$ denotes an unknown bounded function of time. In the case of the existence of a periodic solution, we write

$$\varepsilon y(t) = \varepsilon \sum_{k=2}^{\infty} a_k \sin(k\omega t + \varphi_k) \tag{3}$$

We represent the given nonlinear function $F(x,px)$ in the form

$$F(x,p) = F(x^*,px^*) + [F(x^* + \varepsilon y, px^* + \varepsilon py) - F(x^*,px^*)] \tag{4}$$

Expanding the two components separately in a Fourier series, we obtain

$$F(x,px) = q_0 + \left(A + \frac{B}{\omega} p\right) \sin \omega t + \sum_{k=2}^{\infty} F_k + \varepsilon \sum_{k=0}^{\infty} \Phi_k \tag{5}$$

where q_0, A, and B are the coefficients of the initial term, the sine, and cosine terms, rexpectively, the cosine being replaced by $\frac{1}{\omega} p \sin \omega t$ of the expansion of the function $F(x^*,px^*)$ in a Fourier series. ΣF_k denotes all the higher harmonics of the expansion of $F(x^*,px^*)$ in a Fourier series (they must not be considered small, since the nonlinearity is not small) where we write

$$F_k = b_k \sin (k\omega t + \psi_k) \qquad (k = 2, 3, \ldots, \infty) \tag{6}$$

$\varepsilon \Sigma \Phi_k$ denotes all the terms of the expansion in a Fourier series of all the expressions shown in brackets in formula (4). This entire expression

is written with a small parameter which, according to equation (4), is small if the derivatives $\partial F/\partial x$ and $\partial F/\partial px$ are finite. This expression is also computed as small in the case of certain discontinuous nonlinear characteristics (e.g., the Raleigh type where the preceding derivatives at the points of discontinuity are delta functions). We may write

$$\varepsilon \Phi_k = \varepsilon c_k \sin(k\omega t + \vartheta_k) \qquad (k = 0, 1, 2, \ldots, \infty) \tag{7}$$

We substitute equations (2) and equation (4) in the given equation (1), so that

$$Q(p)x^* + Q(p)\varepsilon y +$$

$$R(p)q_0 + R(p)\left(A + \frac{B}{\omega}p\right)\sin \omega t + R(p)\varepsilon\sum_{k=2}^{\infty}F_k + R(p)\varepsilon\sum_{k=0}^{\infty}\Phi_k = 0 \tag{8}$$

Since the equation must be satisfied identically, we separately equate to zero all coefficients with the same order harmonic. We note that formula (8), in the case of the existence of a periodic solution, is exact.

From the equating of the zeroth harmonics of equation (8), there is obtained with an accuracy up to ε the relation

$$q_0 = \frac{1}{2\pi}\int_0^{2\pi} F(a \sin u, a\omega \cos u)\, du = 0 \qquad (u = \omega t) \tag{9}$$

which is a certain general requirement for $F(x, px)$.

From the equating of the first harmoics of equation (8), taking account of equations (2) and equation (7), we have

$$a \sin \omega t = -\sqrt{A^2 + B^2}\left|\frac{R(i\omega)}{Q(i\omega)}\right|\sin(\omega t + \gamma + \beta) -$$

$$\varepsilon c_1\left|\frac{R(i\omega)}{Q(i\omega)}\right|\sin(\omega t + \vartheta_1 + \beta) \tag{10}$$

where γ and β are arguments of the expressions $A + iB$ and $R(i\omega)/Q(i\omega)$. On the basis of the exact equation (10), we obtain the following approximation (with an accuracy up to ε):

$$a \approx \sqrt{A^2 + B^2}\left|\frac{R(i\omega)}{Q(i\omega)}\right| \qquad \gamma + \beta \approx \pi \tag{11}$$

It is here assumed that the polynomial $Q(p)$ in equation (1) does not have purely imaginary roots.

From the equating of the higher harmonics of equation (8), considering equations (3), (6), and (7), we have

$$\varepsilon a_k \sin(k\omega t + \varphi_k)$$

$$= -b_k \left| \frac{R(ik\omega)}{Q(ik\omega)} \right| \sin(k\omega t + \psi_k + \beta_k) - \varepsilon c_k \left| \frac{R(ik\omega)}{Q(ik\omega)} \right| \sin(k\omega t + \vartheta_k + \beta_k) \quad (12)$$

where β_k denotes the argument of the expression $R(ik\omega)/Q(ik\omega)$.

It is thus seen that if b_k is not small, the magnitude $|R(ik\omega)/Q(ik\omega)|$ should be of the order of ε. The last component in equation (12) will then be of the order ε^2. From the exact equation (12), we obtain the following approximate equation (with an accuracy up to ε):

$$\varepsilon a_k \approx b_k \left| \frac{R(ik\omega)}{Q(ik\omega)} \right| \qquad \psi_k + \beta_k \approx \pi \quad (13)$$

Comparing formulas (13) and (11) we see that, for example, the wish to have in the solution (see eqs. (2) and (3))

$$\sum_{k=2}^{\infty} (\varepsilon a_k)^2 \ll a^2 \quad (14)$$

leads to the need of satisfying, in the given equation (1), the following requirement:

$$\sum_{k=2}^{\infty} b_k^2 \left| \frac{R(ik\omega)}{Q(ik\omega)} \right|^2 \ll (A^2 + B^2) \left| \frac{R(i\omega)}{Q(i\omega)} \right|^2 \quad (15)$$

where its satisfying in the concrete system can be checked after ω is obtained. The degree of $Q(p)$ should, in any case, be higher than that of $R(p)$. A particular case of the general expression (15) is intuitively the earlier introduced "filter property" of the linear part.

Thus, condition (15) has been obtained and must be satisfied by the coefficients of the given differential equation (1) in order that a periodic solution, if it exists, may be approximately determined in the form of sinusoids in the presence of a "strong" nonlinearity of $F(x,px)$.

The equation for its approximate determination according to equation (8) with the substitution of $\sin \omega t = x^*/a$, assumes the form

$$\left[Q(p) + R(p)\left(q + \frac{q_1}{\omega} p\right)\right] x^* = 0 \qquad (16)$$

where

$$q = \frac{A}{a} = \frac{1}{\pi a} \int_0^{2\pi} F(a \sin u, a\omega \cos u) \sin u \, du$$

$$q_1 = \frac{B}{a} = \frac{1}{\pi a} \int_0^{2\pi} F(a \sin u, a\omega \cos u) \cos u \, du \qquad (17)$$

The replacement of equation (1) by equation (16) with its subsequent investigation by the linear methods is called harmonic linearization. This is formally equivalent to following the mode of writing the initial equation (1) with a small parameter, so that

$$Q(p)x + R(p)\left(q + \frac{q_1}{\omega} p\right)x + \varepsilon f(x,p) = 0 \qquad (18)$$

where, according to equation (8), we may write

$$\varepsilon f(x,p) = R(p)\left[\sum_{k=2}^{\infty} F_k + \varepsilon \sum_{k=0}^{\infty} \Phi_k - \left(q + \frac{q_1}{\omega} p\right)\varepsilon y\right]$$

where the first term of the three terms shown in brackets, taken separately, is not small. The smallness of the function $\varepsilon f(x,p)$ is acquired only with the factor $R(p)$ owing to the property of equation (15).

There has thus been found the form of the presence of the small parameter ε in the equations of nonlinear systems of automatic control required for the application of the method of harmonic linearization (this also refers to the first approximation of the method of small parameter which, according to reference 7, coincides with the given method). The writing of equation (1) with small parameter in the form of equation (18) is valid in the region of the existence of equation (2).

For definiteness we shall assume that the polynomials $R(p)$ and $Q(p)$ are such that the characteristic equation (eq. (16)) has all positive coefficients, and that the quotient of the division of the entire left side of equation (16) by $p^2 + \omega^2$ satisfies the criterion of Hurwitz. It may then be said that the given nonlinear system (eq. (1)) for the existence of the periodic solution (eq. (2)) is close to the equivalent linear system which is on the boundary of stability, except for the

circumstance that the linear system is different for various periodic solutions (because of a change in the coefficients q and q_1 from one periodic solution to another, since q and q_1 depend on the amplitude a and the frequency ω of the required solution). Thus, evidently, the equation of the first approximation (eq. (16)) is nonlinear if one speaks of the combination of all possible periodic solutions for different values of the coefficients of the polynomials R and Q (i.e., for different parameters of the system), and is linear only for each given periodic solution (along the given solution). Such is the property of the method of harmonic linearization in its application to systems with "strong" nonlinearity.

By writing the given equation (1) in the form of equation (8) all the variants of the method of harmonic balance in control theory and all generalizations of the method also become easily understandable.

A similar reasoning may also be developed for the application of the method of·N. N. Bogolyubov (ref. 2) to the investigation of transient vibrational processes in certain nonlinear systems of control.

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